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Pathfinder for OLYMPIAD MATHEMATICS

Vikash Tiwari

V. Seshan

Pearson

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MATHEMATICS

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Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128,
formerly known as TutorVista Global Pvt. Ltd, licensee of Pearson Education in South Asia.

ISBN: 9789332568723
eISBN: 9789352862757

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Preface

"For another hundred years, School will teach children 'to do' rather than 'to think'" observed Bertrand Russell. This statement is still seen to be true without being even remotely contradicted.

NCF 2005 (National Curriculum Framework) provides a vision for perspective planning ofschool education in scholastic and non-scholastic domains. It also emphasizes on 'mathematisation' of the child's thought and processes by recognizing mathematics as an integral part of development of the human potential. The higher aim of teaching mathematics is to enhance the ability to visualize, logically understand, build arguments, prove statements and in a sense, handle abstraction. For motivated and talented students, there is a need to widen the horizon as these students love challenges and always look beyond the curriculum at school.

Hence, we created this book to cater to the needs of these students. With numerous problems designed to develop thinking and reasoning, the book contains statements, definitions, postulates, formulae, theorems, axioms, and propositions, which normally do not appear in school textbooks. These are spelt out and interpreted to improve the student's conceptual knowledge.

The book also presents 'non-routine problems' and detailed, step-by-step solutions to these problems to enable the reader to acquire a better understanding of the concepts as well as to develop analytical and reasoning (logical) abilities. Thus, readers get the 'feel' of problem-solving as an activity which, in turn, reveals the innate pleasure of successfully solving a challenging problem. This 'pleasure' is permanent and helps to build-in them a positive attitude towards the subject. Developing ability for critical analysis and problem solving is an essential requirement if one wants to become successful in life.

No one has yet discovered a way of learning mathematics better than, by solving problems in the subject. This book helps students to face competitive examinations such as the Olympiads (RMO, INMO, IMO), KVPY and IIT-JEE confidently without being befuddled by the intricacies of the subject. It has been designed to enable students and all lovers of mathematics to master the subject at their own pace.

We have made efforts to provide solutions along with the problems in an error-free and unambiguous manner as far as possible. However, if any error is detected by the reader, it may please be brought to our notice, so that we may make necessary corrections in the future editions of the book. We look forward to your suggestions and shall be grateful for them.

Lastly, we share the observation made by Pundit Jawaharlal Nehru: "Giving opportunity to potential creativity is a matter of life and death for an enlightened society because the contributions of a few creative individuals are the mankind's ultimate capital asset."

We wish best of luck at all times to all those using this book.

Vikash Tiwari

V. Seshan

Acknowledgements

First and foremost, we thank the Pearson group for motivating us and rendering all possible assistance in bringing out this book in its present form. We are grateful to the Pearson group for having consented to publish the book on our behalf.

We would also like to thank Ajai Lakheena, who has been instrumental in giving this book its present shape. He has made invaluable contributions to "Geometry" chapter of this book.This section would not have been as effective without his efforts.

We also express our gratitude to Bhupinder Singh Tomar and Abir Bhowmick, who have helped us with their discerning inputs and suggestions for making this book error-free. We are indebted to R.K. Thakur for his inputs and constant encouragement to write this book.

This book is dedicated to my wife Priyanka for her kindness, devotion and endless-support in managing household chores and to my two adorable daughters Tanya and Manya who sacrificed their vacation umpteen times for my (our) sake.

Vikash Tiwari

About the Authors

Vikash Tiwari has been teaching students for Mathematical Olympiads (Pre RMO, RMO, INMO and IMOTC) and other examinations like KVPY and JEE *for the last 20 years*. He is a renowned figure in the field of Mathematics across the geography of the country. His students have always founds his methods of teaching insightful and his approach to problem solving very intriguing. He has guided several of the medal winning students that have done India proud at the International Mathematical Olympiad over the years. He use to conduct Olympiad training camps for several organizations such as Kendriya Vidyalaya Sangathan, Delhi Public School (DPS) etc. He has immersed himself into the service of mankind via the medium of Mathematics for the past couple of decades and has come with this first book of his which is basically an inventory of all the concepts required to ace the Mathematics Olympiad at various levels.

V. Seshan is the key resource person shortlisted by CBSE to provide Olympiad training across India. He is a popular teacher and retd. Principal & Director of Bhartiya Vidya Bhavan, Baroda Centre. He is well known for his unique ability in teaching Mathematics with conceptual clarity. With teaching experience spanning over 40 years, he has been instrumental in setting-up the Olympiad Centre in Tata Institute of Fundamental Research, Mumbai. He has also been awarded with various medals, honors and recognitions by prestigious universities and institutions from across the globe. These bear testament to his immense contribution to the field of Mathematics. Many of his students have taken part in both National & International Mathematics Olympiad and have also won gold and silver medals to their credits. A few of his recognition are listed below.

- Fulbright Teacher Awardee (USA-1970)
- Presidential Awardee (1987)
- Advisor to National Science Olympiad Foundation (Since 1989)
- Rotary (Int) Awardee (1992)
- PEE VEE National Awardee (2000)
- Ramanujan Awardee (2008)

Chapter 1

Niccolò Fontana Tartaglia

(1499/1500–13 Dec 1557) Tartaglia was an Italian mathematician. The name "Tartaglia" is actually a nickname meaning "stammerer", a reference to his injury-induced speech impediment. He was largely self-taught, and was the first person to translate Euclid's *Elements* into a modern European language. He is best remembered for his contributions to algebra, namely his discovery of a formula for the solutions to a cubic equation. Such a formula was also found by Gerolamo Cardano at roughly the same time, and the modern formula is known as the Cardano-Tartaglia formula. Cardano also found a solution to the general quartic equation.

Évariste Galois

(25 Oct 1811–31 May 1832) Galois was a very gifted young French mathematician, and his story is one of the most tragic in the history of mathematics. He was killed at the age of 20 in a duel that is still veiled in mystery. Before that, he made huge contributions to abstract algebra. He helped to found group theory as we know it today, and he was the first to use the term "group". Perhaps most importantly, he proved that it is impossible to solve a fifth-degree polynomial (or a polynomial of any higher degree) using radicals by studying permutation groups associated to polynomials. This area of algebra is still important today, and it is known as *Galois theory* in his honor.

Niels Henrik Abel

(5 Aug 1802–6 Apr 1829) Abel was a Norwegian mathematician who, like Galois, did seminal work in algebra before dying at a very young age. Strangely enough, he proved similar results regarding the insolvability of the quintic independently from Galois. In honor of his work in group theory, abelian groups are named after him. The Abel Prize in mathematics, sometimes thought of as the "Nobel Prize in Mathematics," is also named for him.

Joseph-Louis Lagrange

(25 Jan 1736–10 Apr 1813) Despite his French-sounding name, Lagrange was an Italian mathematician. Like many of the great mathematicians of his time, he made contributions to many different areas of mathematics. In particular, he did some early work in abstract algebra.

Polynomials

1.1 Polynomial Functions

Any function, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, is a polynomial function in 'x' where $a_i (i = 0, 1, 2, 3, \ldots, n)$ is a constant which belongs to the set of real numbers and sometimes to the set of complex numbers, and the indices, $n, n-1, \ldots, 1$ are natural numbers. If $a_n \neq 0$, then we can say that $f(x)$ is a polynomial of degree n. a_n is called leading coefficient of the polynomial. If $a_n = 1$, then polynomial is called monic polynomial. Here, if $n = 0$, then $f(x) = a_0$ is a constant polynomial. Its degree is 0, if $a_0 \neq 0$. If $a_0 = 0$, the polynomial is called zero polynomial. Its degree is defined as $-\infty$ to preserve the first two properties listed below. Some people prefer not to defined degree of zero polynomial. The domain and range of the function are the set of real numbers and complex numbers, respectively. Sometimes, we take the domain also to be complex numbers. If z is a complex number and $f(z) = 0$, then z is called 'a zero of the polynomial'.

If $f(x)$ is a polynomial of degree n and $g(x)$ is a polynomial of degree m then

1. $f(x) \pm g(x)$ is polynomial of degree $\leq \max\{n, m\}$
2. $f(x) \cdot g(x)$ is polynomial of degree $m + n$
3. $f(g(x))$ is polynomial of degree $m \cdot n$, where $g(x)$ is a non-constant polynomial.

Illustrations

1. $x^4 - x^3 + x^2 - 2x + 1$ *is a polynomial of degree* 4 and 1 *is a zero of the polynomial as*
$$1^4 - 1^3 + 1^2 - 2 \times 1 + 1 = 0.$$
2. $x^3 - ix^2 + ix + 1 = 0$ *is a polynomial of degree* 3 and i *is a zero of his polynomial as* $i^3 - i \cdot i^2 + i \cdot i + 1 = -i + i - 1 + 1 = 0$.
3. $x^2 - (\sqrt{3} - \sqrt{2})x - \sqrt{6}$ is *a polynomial of degree* 2 *and* $\sqrt{3}$ *is a zero of this polynomial as* $(\sqrt{3})^2 - (\sqrt{3} - \sqrt{2})\sqrt{3} - \sqrt{6} = 3 - 3 + \sqrt{6} - \sqrt{6} = 0$.

Note: The above-mentioned definition and examples refer to polynomial functions in one variable. Similarly, polynomials in 2, 3, …, n variables can be defined. The domain for polynomial in n variables being the set of (ordered) n tuples of complex numbers and the range is the set of complex numbers.

Illustration *$f(x, y, z) = x^2 - xy + z + 5$ is a polynomial in x, y, z of degree 2 as both x^2 and xy have degree 2 each.*

Note: In a polynomial in n variables, say, $x_1, x_2, \ldots, x_n$, a general term is $x_1^{k_1} \cdot x_2^{k_2} \cdots x_n^{k_n}$. Degree of the term is $k_1 + k_2 + \cdots + k_n$ where $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, n$. The degree of a polynomial in n variables is the maximum of the degrees of its terms.

1.2 Division in Polynomials

If $P(x)$ and $\phi(x)$ ($\phi(x) \not\equiv 0$) are any two polynomials, then we can find unique polynomials $Q(x)$ and $R(x)$, such that $P(x) = \phi(x) \times Q(x) + R(x)$ where the degree of $R(x) <$ degree of $\phi(x)$ or $R(x) \equiv 0$. $Q(x)$ is called the quotient and $R(x)$, the remainder.

In particular, if $P(x)$ is a polynomial with complex coefficients, and a is a complex number, then there exists a polynomial $Q(x)$ of degree 1 less than $P(x)$ and a complex number R, such that $P(x) = (x - a)Q(x) + R$.

Illustration $x^5 = (x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4) + a^5$.

Here, $P(x) = x^5$, $Q(x) = x^4 + ax^3 + a^2x^2 + a^3x + a^4$,

and $R = a^5$.

Example 1 *What is the remainder when $x + x^9 + x^{25} + x^{49} + x^{81}$ is divided by $x^3 - x$.*

Solution: We have,

$$x + x^9 + x^{25} + x^{49} + x^{81} = x(1 + x^8 + x^{24} + x^{48} + x^{80})$$
$$= x[(x^{80} - 1) + (x^{48} - 1) + (x^{24} - 1) + (x^8 - 1) + 5]$$
$$= x(x^{80} - 1) + x\,(x^{48} - 1) + x(x^{24} - 1) + x(x^8 - 1) + 5x$$

Now, $x^3 - x = x(x^2 - 1)$ and all, but the last term $5x$ is divisible by $x(x^2 - 1)$. Thus, the remainder is $5x$.

Example 2 *Prove that the polynomial $x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$ is divisible by $x^9 + x^8 + x^7 + \cdots + x + 1$.*

Solution: Let,

$$P = x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$$
$$Q = x^9 + x^8 + x^7 + \cdots + x + 1$$
$$P - Q = x^9(x^{9990} - 1) + x^8(x^{8880} - 1) + x^7(x^{7770} - 1) + \cdots + x(x^{1110} - 1)$$
$$= x^9[(x^{10})^{999} - 1] + x^8[(x^{10})^{888} - 1] + x^7[(x^{10})^{777} - 1] + \cdots + x[(x^{10})^{111} - 1] \qquad (1)$$

But, $(x^{10})^n - 1$ is divisible by $x^{10} - 1$ for all $n \geq 1$.

$\therefore$ RHS of Eq. (1) divisible by $x^{10} - 1$.

$\therefore$ $P - Q$ is divisible by $x^{10} - 1$

As $x^9 + x^8 + \cdots + x + 1 \mid (x^{10} - 1)$

$\Rightarrow$ $x^9 + x^8 + x^7 + \cdots + x + 1 \mid (P - Q)$

$\Rightarrow$ $x^9 + x^8 + x^7 + \cdots + x + 1 \mid P$

1.3 Remainder Theorem and Factor Theorem

1.3.1 Remainder Theorem

If a polynomial $f(x)$ is divided by $(x - a)$, then the remainder is equal to $f(a)$.

Proof:

$$f(x) = (x - a)Q(x) + R$$

and so, $$f(\mathrm{a}) = (a - a)Q(a) + R = R.$$

If $R = 0$, then $f(x) = (x - a)Q(x)$ and hence, $(x - a)$ is a factor of $f(x)$.

Further, $f(a) = 0$, and thus, a is a zero of the polynomial $f(x)$. This leads to the factor theorem.

1.3.2 Factor Theorem

$(x - a)$ is a factor of polynomial $f(x)$, if and only if, $f(a) = 0$.

Example 3 *If $f(x)$ is a polynomial with integral coefficients and, suppose that $f(1)$ and $f(2)$ both are odd, then prove that there exists no integer n for which $f(n) = 0$.*

Solution: Let us assume the contrary. So, $f(n) = 0$ for some integer n.

Then, $(x - n)$ divides $f(x)$.

Therefore, $f(x) = (x - n)g(x)$

where $g(x)$ is again a polynomial with integral coefficients.

Now, $f(1) = (1 - n)\, g(1)$ and $f(2) = (2 - n)\, g(2)$ are odd numbers but one of $(1 - n)$ and $(2 - n)$ should be even as they are consecutive integers.

Thus one of $f(1)$ and $f(2)$ should be even, which is a contradiction. Hence, the result.

Aliter: See the Example (41) on page 6.24 in Number Theory chapter.

Example 4 *If f is a polynomial with integer coefficients such that there exists four distinct integer a_1, a_2, a_3 and a_4 with $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$, show that there exists no integer b, such that $f(b) = 1993$.*

Solution: Suppose, there exists an integer b, such that $f(b) = 1993$, let $g(x) = f(x) - 1991$.

Now, g is a polynomial with integer coefficients and $g(a_i) = 0$ for $i = 1, 2, 3, 4$.

Thus $(x - a_1)(x - a_2)(x - a_3)$ and $(x - a_4)$ are all factors of $g(x)$.

So, $g(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \times h(x)$

where $h(x)$ is polynomial with integer coefficients.

$$\begin{aligned} g(b) &= f(b) - 1991 \\ &= 1993 - 1991 = 2 \text{ (by our choice of } b) \end{aligned}$$

But, $g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)\, h(b) = 2$

Thus, $(b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are all divisors of 2 and are distinct.

$\therefore$ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are $1, -1, 2, -2$ in some order, and $h(b)$ is an integer.

$\therefore$ $g(b) = 4 \cdot h(b) \neq 2$.

Hence, such b does not exist.

1.4 Fundamental Theorem of Algebra

Every polynomial function of degree ≥ 1 has at least one zero in the complex numbers. In other words, if we have

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $n \geq 1$, then there exists atleast one $h \in \mathbb{C}$, such that

$$a_n h^n + a_{n-1} h^{n-1} + \cdots + a_1 h + a_0 = 0.$$

From this, it is easy to deduce that a polynomial function of degree 'n' has exactly n zeroes.
i.e., $f(x) = a(x - r_1)(x - r_2)\ldots(x - r_n)$

Notes:

1. Some of the zeroes of a polynomial may repeat.
2. If a root α is repeated m times, then m is called multiplicity of the root 'α' or α is called m fold root.
3. The real numbers of the form $\sqrt{3}$, $\sqrt{5}$, $\sqrt{12}$, $\sqrt{27}$, …, $\sqrt{5}+\sqrt{3}$, etc. are called, 'quadratic surds'. In general, $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{a}+\sqrt{b}$, etc. are quadratic surds, if a, b are not perfect squares. In a polynomial with integral coefficients (or rational coefficients), if one of the zeroes is a quadratic surd, then it has the conjugate of the quadratic surd also as a zero.

Illustration *$f(x) = x^2 + 2x + 1 = (x + 1)^2$ and the zeroes of $f(x)$ are -1 and -1. Here, it can be said that $f(x)$ has a zero -1 with multiplicity two.*

Similarly, $f(x) = (x + 2)^3(x - 1)$ has zeroes $-2, -2, -2, 1$, i.e., the zeroes of $f(x)$ are -2 with multiplicity 3 and 1.

Example 5 *Find the polynomial function of lowest degree with integral coefficients with $\sqrt{5}$ as one of its zeroes.*

Solution: Since the order of the surd $\sqrt{5}$ is 2, you may expect that the polynomial of the lowest degree to be a polynomial of degree 2.

Let, $\qquad P(x) = ax^2 + bx + c;\ a, b, c \in \mathbb{Q}$

$$P(\sqrt{5}) = 5a + \sqrt{5}b + c = 0 \quad \Rightarrow \quad (5a + c) + \sqrt{5}b = 0$$

But, $\sqrt{5}$ is irrational.
So,

$$5a + c = 0 \quad \text{and} \quad b = 0$$

$$\Rightarrow \quad c = -5a \quad \text{and} \quad b = 0.$$

So, the required polynomial function is $P(x) = ax^2 - 5a,\ a \in \mathbb{Z}\setminus\{0\}$

You can find the other zero of this polynomial to be $-\sqrt{5}$.

Aliter: You know that any polynomial function having, say, n zeroes $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be written as $P(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$ and clearly, this function is of nth degree. Here, the coefficients may be rational, real or complex depending upon the zeroes $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If the zero of a polynomial is $\sqrt{5}$, then $P_0(x) = (x - \sqrt{5})$ or $a(x - \sqrt{5})$.
But, we want a polynomial with rational coefficients.
So, here we multiply $(x - \sqrt{5})$ by the conjugate of $x - \sqrt{5}$, *i.e.*, $x + \sqrt{5}$. Thus, we get the polynomial $P(x) = (x - \sqrt{5})(x + \sqrt{5})$, where the other zero of $P(x)$ is $-\sqrt{5}$.

Now, $P_1(x) = x^2 - 5$, with coefficient of $x^2 = 1$, $x = 0$ and constant term -5, and all these coefficients are rational numbers.

Now, we can write the required polynomial as $P(x) = ax^2 - 5a$ where a is a non-zero integer.

Example 6 *Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is* $\sqrt{5}+\sqrt{2}$.

Solution: Let, $P(x) = x - (\sqrt{5}+\sqrt{2}) = [(x-\sqrt{5})-\sqrt{2}]$.

Now, following the method used in the previous example, using the conjugate, we get:

$$P_1(x) = [(x-\sqrt{5})-\sqrt{2}][(x-\sqrt{5})+\sqrt{2}]$$
$$= (x^2 - 2\sqrt{5}x + 5) - 2$$
$$= (x^2 + 3 - 2\sqrt{5}x)$$
$$P_2(x) = [(x^2+3)-2\sqrt{5}x][(x^2+3)+2\sqrt{5}x]$$
$$= (x^2 + 3)^2 - 20x^2$$
$$= x^4 + 6x^2 + 9 - 20x^2$$
$$= x^4 - 14x^2 + 9$$
$$P(x) = ax^4 - 14ax^2 + 9a, \quad \text{where } a \in \mathbb{Z}, a \neq 0.$$

The other zeroes of this polynomial are $\sqrt{5}-\sqrt{2}, -\sqrt{5}+\sqrt{2}, -\sqrt{5}-\sqrt{2}$.

1.4.1 Identity Theorem

A polynomials $f(x)$ of degree n is identically zero if it vanishes for atleast $n + 1$ distinct values of 'x'.

Proof: Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be n distinct values of x at which $f(x)$ becomes zero.
Then we have

$$f(x) = a(x - x_1)(x - x_2)\ldots(x - x_n)$$

Let α_{n+1} be the $n+1^{\text{th}}$ value of x at which $f(x)$ vanishes. Then

$$f(\alpha_{n+1}) = a(\alpha_{n+1} - \alpha_1)(\alpha_{n+1} - \alpha_2)\ldots(\alpha_{n+1} - \alpha_n) = 0$$

As α_{n+1} is different from $\alpha_1, \alpha_2 \ldots \alpha_n$ none of the number $\alpha_{n+1} - \alpha_i$ vanishes for $i = 1, 2, 3, \ldots n$. Hence $a = 0 \Rightarrow f(x) \equiv 0$.
Using above result we can say that,
If two polynomials $f(x)$ and $g(x)$ of degree m, n respectively with $m \leq n$ have equal values at $n + 1$ distinct values of x, then they must be equal.

Proof: Let $P(x) = f(x) - g(x)$, now degree of $P(x)$ is at most 'n' and it vanishes for at least $n + 1$ distinct values of $x \Rightarrow P(x) \equiv 0 \Rightarrow f(x) \equiv g(x)$.

Corollary: The only periodic polynomial function is constant function.
i.e., if $f(x)$ is polynomials with $f(x + T) = f(x)\ \forall\ x \in \mathbb{R}$ for some constant T then $f(x) =$ constant $= c$ (say)

Proof: Let $f(0) = c$
$\Rightarrow f(0) = f(T) = f(2T) = \ldots = c$
$\Rightarrow$ Polynomial $f(x)$ and constant polynomial $g(x) = c$ take same values at an infinite number of points. Hence they must be identical.

Example 7 Let $P(x)$ be a polynomial such that $x \cdot P(x-1) = (x-4)\,P(x)\ \forall\, x \in \mathbb{R}$. Find all such $P(x)$.

Solution: Put $x = 0$, $0 = -4\,P(0)$

$\Rightarrow P(0) = 0$

Put $x = 1$, $1 \cdot P(0) = -3\,P(1)$

$\Rightarrow P(1) = 0$

Put $x = 2$, $2 \cdot P(1) = -2\,P(2)$

$\Rightarrow P(2) = 0$

Put $x = 3$, $3 \cdot P(2) = -P(3)$

$\Rightarrow P(3) = 0$

Let us assume $P(x) = x(x-1)(x-2)(x-3)\,Q(x)$, where $Q(x)$ is some polynomial.
Now using given relation we have

$$x(x-1)(x-2)(x-3)(x-4)Q(x-1) = x(x-1)(x-2)(x-3)(x-4)Q(x)$$

$$\Rightarrow \quad Q(x-1) = Q(x) \quad \forall x \in \mathbb{R} - \{0,1,2,3,4\}$$

$$\Rightarrow \quad Q(x-1) = Q(x) \quad \forall x \in \mathbb{R} \quad \text{(From identity theorem)}$$

$$\Rightarrow \quad Q(x) \text{ is periodic}$$

$$\Rightarrow \quad Q(x) = c$$

$$\Rightarrow \quad P(x) = cx(x-1)(x-2)(x-3)$$

Example 8 Let $P(x)$ be a monic cubic equation such that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, then find $P(4)$.

Solution: as $P(x)$ is a monic, coefficient of highest degree will be '1'.
Let $Q(x) = P(x) - x$, where $Q(x)$ is also monic cubic polynomial.

$$Q(1) = P(1) - 1 = 0;\ Q(2) = P(2) - 2 = 0;\ Q(3) = P(3) - 3 = 0$$

$$\Rightarrow \quad Q(x) = (x-1)(x-2)(x-3)$$

$$\Rightarrow \quad P(x) = Q(x) + x = (x-1)(x-2)(x-3) + x$$

$$\Rightarrow \quad P(x) = (4-1)(4-2)(4-3) + 4 = 10$$

Build-up Your Understanding 1

1. Find a fourth degree equation with rational coefficients, one of whose roots is, $\sqrt{3} + \sqrt{7}$.
2. Find a polynomial equation of the lowest degree with rational coefficients whose one root is $\sqrt[3]{2} + 3\sqrt[3]{4}$.
3. Form the equation of the lowest degree with rational coefficients which has $2 + \sqrt{3}$ and $3 + \sqrt{2}$ as two of its roots.
4. Show that $(x-1)^2$ is a factor of $x^n - nx + n - 1$.
5. If a, b, c, d, e are all zeroes of the polynomial $(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1)$, find the value of $(1+a)(1+b)(1+c)(1+d)(1+e)$.
6. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the roots of the equation $x^n - 1 = 0$, $n \in \mathbb{N}$, $n \geq 2$ show that $n = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \ldots (1-\alpha_{n-1})$.
7. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, show that $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = (1-q+s)^2 + (p-r)^2$.

8. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is a polynomial such that $f(1) = 10, f(2) = 20, f(3) = 30$, find the value of $\dfrac{f(12)+f(-8)}{10}$. **[CMO, 1984]**

9. The polynomial $x^{2k} + 1 + (x + 1)^{2k}$ is not divisible by $x^2 + x + 1$. Find the value of $k \in \mathbb{N}$.

10. Find all polynomials $P(x)$ with real coefficients such that

$$(x - 8)P(2x) = 8(x - 1)P(x).$$

11. Let $(x - 1)^3$ divides $(p(x) + 1)$ and $(x + 1)^3$ divides $(p(x)-1)$. Find the polynomial $p(x)$ of degree 5.

1.5 Polynomial Equations

Let, $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$; $a_n \neq 0$, $n \geq 1$ be a polynomial function.

Then, $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ is called a polynomial equation in x of degree n. Thus,

1. Every polynomial equation of degree n has n roots counting repetition.
2. If $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ (1)

$a_n \neq 0$ and a_i, $(i = 0, 1, 2, 3, \ldots, n)$ are all real numbers and if, $\alpha + i\beta$ is a zero of (1), then $\alpha - i\beta$ is also a root. For real polynomial, complex roots occur in conjugate pairs.

However, if the coefficients of Eq. (1) are complex numbers, it is not necessary that the roots occur in conjugate pairs.

Example 9 *Form a polynomial equation of the lowest degree with* $3 + 2i$ *and* $2 + 3i$ *as two of its roots, with rational coefficients.*

Solution: Since, $3 + 2i$ and $2 + 3i$ are roots of polynomial equation with rational coefficients, $3 - 2i$ and $2 - 3i$ are also the roots of the polynomial equation. Thus, we have identified four roots so that there are 2 pairs of roots and their conjugates. So, the lowest degree of the polynomial equation should be 4. The polynomial equation should be

$$P(x) = a\,[x-(3-2i)][x-(3+2i)][x-(2+3i)]\,[x-(2-3i)] = 0$$
$$\Rightarrow \quad a\,[(x-3)^2+4][(x-2)^2+9] = 0$$
$$\Rightarrow \quad a\,((x-3)^2(x-2)^2 + 9(x-3)^2 + 4(x-2)^2 + 36) = 0$$
$$\Rightarrow \quad a\,((x^2-5x+6)^2 + 9(x^2-6x+9) + 4(x^2-4x+4) + 36) = 0$$
$$\Rightarrow \quad a\,(x^4 - 10x^3 + 50x^2 - 130x + 169) = 0, \quad a \in \mathbb{Q}\backslash\{0\}$$

1.5.1 Rational Root Theorem

An important theorem regarding the rational roots of polynomial equations:

If the rational number $\dfrac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, $gcd(p,q) = 1$, *i.e.*, p and q are relatively prime, is a root of the equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are integers and $a_n \neq 0$, then p is a divisor of a_0 and q that of a_n.

Proof: Since $\dfrac{p}{q}$ is a root, we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\frac{p}{q} + a_0 = 0$$

$$\Rightarrow \quad a_n p^n + a_{n-1}qp^{n-1} + \cdots + a_1 q^{n-1}p + a_0 q^n = 0 \tag{1}$$

$$\Rightarrow \quad a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1 q^{n-2}p + a_0 q^{n-1} = -\frac{a_n p^n}{q} \tag{2}$$

Since the coefficients $a_{n-1}, a_{n-2}, \ldots, a_0$ and p, q are all integers, hence the left-hand side is an integer, so the right- hand side is also an integer. But, p and q are relatively prime to each other, therefore q should divide a_n.

Again,

$$a_n p^n + a_{n-1}qp^{n-1} + \cdots + a_1 q^{n-1}p = a_0 q^n$$

$$\Rightarrow \quad a_n p^{n-1} + a_{n-1}qp^{n-2} + \cdots + a_1 q^{n-1} = \frac{a_0 q^n}{p} \tag{3}$$

$$\Rightarrow \quad p \mid a_0$$

As a consequence of the above theorem, we have the following corollary.

1.5.2 Corollary (Integer Root Theorem)

Every rational root of $x^n + a_{n-1}x^{n-1} + \cdots + a_0; 0 \le i \le n-1$ is an integer, where $a_i (i = 0, 1, 2, \ldots, n-1)$ is an integer, and each of these roots is a divisor of a_0.

Example 10 *Find the roots of the equation* $x^4 + x^3 - 19x^2 - 49x - 30$, *given that the roots are all rational numbers.*

Solution: Since all the roots are rational by the above corollary, they are the divisors of -30.

The divisors of -30 are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$.

By applying the remainder theorem, we find that $-1, -2, -3$, and 5 are the roots.

Hence, the roots are $-1, -2, -3$ and $+5$.

Example 11 *Find the rational roots of* $2x^3 - 3x^2 - 11x + 6 = 0$.

Solution: Let the roots be of the form $\frac{p}{q}$, where $(p, q) = 1$ and $q > 0$.

Then, since $q/2$, q must be 1 or 2

and $p|6 \quad \Rightarrow \quad p = \pm1, \pm2, \pm3, \pm6$

By applying the remainder theorem,

$$f\left(\frac{1}{2}\right) = f\left(\frac{-2}{1}\right) = f\left(\frac{3}{1}\right) = 0.$$

(Corresponding to $q = 2$ and $p = 1$; $q = 1, p = -2$; $q = 1, p = 3$, respectively.)

So, the three roots of the equation are $\frac{1}{2}$, -2, and 3.

Example 12 *Solve:* $x^3 - 3x^2 + 5x - 15 = 0$.

Solution: $x^3 - 3x^2 + 5x - 15 = 0 \quad \Rightarrow \quad (x^2 + 5)(x - 3) = 0$

$$\Rightarrow \quad x = \pm\sqrt{5}\, i, 3.$$

So the solution are 3, $\sqrt{5}\, i, -\sqrt{5}\, i$.

Example 13 *Show that* $f(x) = x^{1000} - x^{500} + x^{100} + x + 1 = 0$ *has no rational roots.*

Solution: If there exists a rational root, let it be $\frac{p}{q}$ where $(p, q) = 1$, $q \neq 0$. Then, q should divide the coefficient of the leading term and p should divide the constant term.

Thus, $q \mid 1 \quad \Rightarrow \quad q = \pm 1$,

And $p \mid 1 \quad \Rightarrow \quad p = \pm 1$

Thus, $\frac{p}{q} = \pm 1$

If the root $\frac{p}{q} = 1$,

Then, $f(1) = 1 - 1 + 1 + 1 + 1 = 3 \neq 0$,

so, 1 is not a root.

If $\frac{p}{q} = -1$,

Then, $f(-1) = 1 - 1 + 1 - 1 + 1 = 1 \neq 0$

And hence, (-1) is not a root.

Thus, there exists no rational roots for the given polynomial.

1.6 Vieta's Relations

Francois Viète

1540–23 Feb 1603

Nationality: French

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the polynomial equation

$$a_n x_n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0 \quad (a_n \neq 0),$$

then,

$$\sum_{1 \le i \le n} \alpha_i = -\frac{a_{n-1}}{a_n}; \sum_{1 \le i < j \le n} \alpha_i \cdot \alpha_j = \frac{a_{n-2}}{a_n}$$

$$\sum_{1 \le i < j < k \le n} \alpha_i \alpha_j \alpha_k = -\frac{a_{n-3}}{a_n}, \cdots; \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

If we represent the sum $\sum\alpha_i$, $\sum\alpha_i\alpha_j$, …, $\sum\alpha_i\alpha_j \ldots \alpha_n$, respectively, as $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$, (Read it 'sigma 1', 'sigma 2', etc.)

then,

$$\sigma_1 = -\frac{a_{n-1}}{a_n}, \sigma_2 = \frac{a_{n-2}}{a_n}, \ldots$$

$$\sigma_r = (-1)^r \frac{a_{n-r}}{a_n}, \ldots, \sigma_n = (-1)^n \frac{a_0}{a_n}$$

These relations are known as Vieta's relations.

Let us consider the following quadratic, cubic and biquadratic equations and see how we can relate $\sigma_1, \sigma_2, \sigma_3, \ldots,$ with the coefficients.